PURELY INFINITE, SIMPLE C^*-ALGEBRAS ARISING FROM FREE PRODUCT CONSTRUCTIONS, II

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Abstract. Certain reduced free products of C*–algebras with respect to faithful states are simple and purely infinite.

Introduction.

Given unital C*–algebras A and B with states ϕ_A and ϕ_B, whose GNS representation are faithful, their reduced free product C*–algebra,

$$(\mathfrak{A},\phi)=(A,\phi_A)*(B,\phi_B), \tag{1}$$

was introduced in [**12**] and [**1**]. It is the natural construction in Voiculescu's free probability theory (see [**13**]), and Voiculescu's theory has been vital to the study of these C*–algebras.

In [**1**], D. Avitzour showed that reduced free product C*–algebra $\mathfrak{A}$ in (1) is simple if A and B are not too small (with respect to their states), in that they have enough orthogonal unitaries.

A unital C*–algebra is said to be *infinite* if it contains a nonunitary isometry, and is said to be *purely infinite* if every hereditary C*–subalgebra of it is infinite. One of the most interesting open questions about simple C*–algebras is whether every infinite simple C*–algebra is purely infinite. In [**9**], M. Rørdam and the author showed that in the free product (1), if ϕ_A and ϕ_B are faithful, if one of them is not a trace and if A and B are not too small in that they satisfy a condition like (but slightly weaker than) Avitzour's condition, then the free product C*–algebra $\mathfrak{A}$ is infinite, and furthermore, it is properly infinite. It remained open whether these C*–algebras are purely infinite, or indeed whether any C*–algebras arising as reduced free products with respect to faithful states are purely infinite. (In [**8**], the same authors had

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shown that some C*–algebras arising as reduced free product with respect to nonfaithful states are purely infinite.)

In this paper, we show that certain free product C*–algebras $\mathfrak{A}$ in (1), with respect to faithful states, are simple and purely infinite. For example, if $A = B = M_2(\mathbf{C})$ and if ϕ_A and ϕ_B are faithful states on $M_2(\mathbf{C})$ that are not unitarily equivalent then $\mathfrak{A}$ is simple and purely infinite. Note that it was previously not known whether any of these particular reduced free product C*–algebras of $M_2(\mathbf{C})$ with $M_2(\mathbf{C})$ were even simple. (See Examples 3.2 for some more examples.)

The most striking condition that we require of (A, ϕ_A) and (B, ϕ_B) for the free product C*–algebra (1) to be purely infinite and simple is that one of the algebras, say A, contain a partial isometry, v, with orthogonal domain and range projections and scaling the state ϕ_A by λ for some $0 < \lambda < 1$, i.e. that $\phi_A(va) = \lambda\phi_A(av)$ for every $a \in A$. These are fairly strong conditions, but they do arise in a large number of situations. Although both the statement and proof of the main result, Theorem 3.1, are quite technical, we believe they comprise an important advance in the understanding of simplicity and infiniteness of reduced free product C*–algebras.

§1. Notation.

Let us briefly describe some notation.

1.1. Given a unital C*–algebra A and a state, ψ of A, (which will usually be implicit from the context), we use the symbol A^{o} to denote the kernel of ψ.

1.2. If A is a C*–algebra with a state ϕ, then for any C*–subalgebra, $D \subseteq A$, we define

$$A \ominus D = \{a \in A \mid \forall d \in D,\, \phi(da) = 0\}.$$

This is just the orthocomplement of D in the Hilbert space of the GNS representation, pulled back to A.

1.3. If X_ι $(\iota \in I)$ are subsets of an algebra A, then

$$\Lambda^{\mathrm{o}}\big((X_\iota)_{\iota\in I}\big) = \{x_1x_2\cdots x_n \mid n \in \mathbf{N},\, x_j \in X_{\iota_j},\, \iota_1 \neq \iota_2, \iota_2 \neq \iota_3, \dots, \iota_{n-1} \neq \iota_n\},$$

(written simply $\Lambda^{\mathrm{o}}(X_1, X_2)$ if $I = \{1, 2\}$), is the set of all "alternating words" in the X_ι.

§2. Conditional expectations.

Tomiyama [**11**] proved that if $B \subseteq A$ is a C*–subalgebra and if $E : A \to B$ is a projection of norm 1 then E is positive and a conditional expectation. In this section, we consider the reduced free product of conditional expectations, a special case of which will be important in the sequel. The results proved here are perhaps well known, but their proofs are included out of a desire for completeness. This section can be viewed as a C*–version of [**4**, §3].

Lemma 2.1. *Let A be a unital C^*–algebra and let $B \subseteq A$ be a unital C^*–subalgebra. Let ϕ be a faithful state on A and let $(\pi, \mathcal{H}, \xi) = \mathrm{GNS}(A, \phi)$, so that $\{\hat{a} \mid a \in A\}$ is a dense subspace of $\mathcal{H}$ with inner product $\langle \hat{a}_1, \hat{a}_2 \rangle = \phi(a_2^* a_1)$. Let $\mathcal{H}_B = \overline{\{\hat{b} \mid b \in B\}} \subseteq \mathcal{H}$ and let P be the projection from $\mathcal{H}$ onto $\mathcal{H}_B$. Suppose that for some norm–dense subset, X, of A, we have $P\pi(x)|_{\mathcal{H}_B} \in P\pi(B)|_{\mathcal{H}_B}$ for every $x \in X$. Then there is a projection, E, of norm 1 from A onto B, satisfying $\phi \circ E = \phi$.*

Proof. Since ϕ is faithful, π is a faithful representation of A. Let $\pi_B : B \to \mathcal{B}(\mathcal{H}_B)$ be $\pi_B(b) = P\pi(b)|_{\mathcal{H}_B}$. Then $(\pi_B, \mathcal{H}_B, \xi) = \mathrm{GNS}(B, \phi|_B)$, and since $\phi|_B$ is faithful on B, π_B is a faithful representation of B. Now from $P\pi(X)|_{\mathcal{H}_B} \subseteq \pi_B(B)$ and taking limits in norm we get that $P\pi(A)|_{\mathcal{H}_B} \subseteq \pi_B(B)$, and hence we may define $E : A \to B$ by $E(a) = \pi_B^{-1}(P\pi(a)|_{\mathcal{H}_B})$. Then clearly $E(b) = b$ if $b \in B$ and $\|E(a)\| \le \|a\|$ for every $a \in A$, so E is a projection of norm 1. Moreover,

$$\phi(E(a)) = \langle \pi_B(E(a))\xi, \xi \rangle = \langle P\pi(a)\xi, \xi \rangle = \langle \pi(a)\xi, \xi \rangle = \phi(a),$$

so E preserves ϕ.

□

Let us here recall (see Lemma 3.2 of [**4**]) that the converse of the above lemma is true, namely, if B is a C*–subalgebra of the C*–algebra A, and if $E : A \to B$ is a projection of norm 1 such that $\phi \circ E = \phi$ for a faithful state ϕ, on A, then E is implemented by a projection P in Hilbert space of the GNS representation.

Lemma 2.2. *Let A be a C^*–algebra with a faithful state ϕ and let $E : A \to B$ be a projection of norm 1 onto a C^*–subalgebra, B, of A. Suppose $\phi \circ E = \phi$. Then*

$$\ker E = A \ominus B \tag{2}$$

and

$$A = (A \ominus B) + B. \tag{3}$$

Proof. Let $(\pi, \mathcal{H}, \xi) = \mathrm{GNS}(A, \phi)$ and let P be the projection from $\mathcal{H}$ onto $\mathcal{H}_B = \{\hat{b} \mid b \in B\}$. Then by [**4**, 3.2], $E(a)\hat{\ } = P\hat{a}$. This shows that $\ker E \subseteq A \ominus B$. To show the opposite inclusion, suppose $a \in A \ominus B$. Then $0 = E(E(a)^* a) = E(a)^* E(a)$, so $E(a) = 0$. Now (3) follows from (2). $\square$

Proposition 2.3. *Suppose A_ι are unital C^*–algebras with faithful states ϕ_ι, for ι in some index set I. Suppose $B_\iota \subseteq A_\iota$ are unital C^*–subalgebras with projections of norm 1, $E_\iota : A_\iota \to B_\iota$, such that $\phi_\iota \circ E_\iota = \phi_\iota$. Let $(\mathfrak{A}, \phi) = \mathop{*}\limits_{\iota \in I} (A_\iota, \phi_\iota)$ be the free product of C^*–algebras and consider the C^*–subalgebra $\mathfrak{B} = C^*(\bigcup_{\iota \in I} B_\iota) \subseteq \mathfrak{A}$. Then there is a projection of norm 1, $E : \mathfrak{A} \to \mathfrak{B}$ such that $\phi \circ E = \phi$ and $E(a) = E_\iota(a)$ whenever $a \in A_\iota$.*

Proof. Let $(\pi_\iota, \mathcal{H}_\iota, \xi_\iota) = \mathrm{GNS}(A_\iota, \phi_\iota)$ and $(\pi, \mathcal{H}, \xi) = \mathrm{GNS}(\mathfrak{A}, \phi)$. From the free product construction we have that

$$\mathcal{H} = \mathbf{C}\xi \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n}} \mathcal{H}^{\circ}_{\iota_1} \otimes \cdots \otimes \mathcal{H}^{\circ}_{\iota_n},$$

where $\mathcal{H}^{\circ}_\iota = \mathcal{H}_\iota \ominus \mathbf{C}\xi_\iota$. By [**4**, Lemma 3.2], for each $\iota \in I$ there is a projection $P_\iota : \mathcal{H}_\iota \to \mathcal{H}_{B_\iota} \overset{\text{def}}{=} \{\hat{b} \mid b \in B_\iota\}$ such that for every $a \in A_\iota$, $P_\iota \pi_\iota(a)|_{\mathcal{H}_{B_\iota}} = P_\iota \pi_\iota(E_\iota(a))|_{\mathcal{H}_{B_\iota}}$. Now $Y \overset{\text{def}}{=} \mathrm{span}\big(\{1\} \cup \Lambda^{\circ}((B^{\circ}{}_\iota)_{\iota \in I})\big)$ is a dense subset of B, so

$$\mathcal{H}_B \overset{\text{def}}{=} \overline{\{\hat{b} \mid b \in B\}} = \overline{\{\hat{b} \mid b \in Y\}} = \mathbf{C}\xi \oplus \bigoplus_{\substack{n \geq 1 \\ \iota_1 \neq \iota_2 \neq \cdots \neq \iota_n}} \mathcal{H}^{\circ}_{B_{\iota_1}} \otimes \cdots \otimes \mathcal{H}^{\circ}_{B_{\iota_n}},$$

where $\mathcal{H}^{\circ}_{B_\iota} = \mathcal{H}_{B_\iota} \ominus \mathbf{C}\xi_\iota$. Let $P : \mathcal{H} \to \mathcal{H}_B$ be the projection onto $\mathcal{H}_B$. Now $X \overset{\text{def}}{=} \mathrm{span}(\{1\} \cup \Lambda^{\circ}((A^{\circ}{}_\iota)_{\iota \in I}))$ is a dense subset of A, and (from the free product construction) we see that

$$P\pi(X)|_{\mathcal{H}_B} \subseteq P\pi(B)|_{\mathcal{H}_B},$$

so by Lemma 2.1, there is a projection of norm 1, $E : A \to B$, satisfying $\phi \circ E = \phi$ and given by $P\pi(E(a))|_{\mathcal{H}_B} = P\pi(a)|_{\mathcal{H}_B}$. From this and the free product construction, we see that $E(a) = E_\iota(a)$ whenever $a \in A_\iota \subseteq A$. $\square$

Definition 2.4. The projection of norm 1, E, found in the above proposition is called the *free product* of the E_ι, and is denoted $E = \mathop{*}\limits_{\iota \in I} E_\iota$.

Corollary 2.5. *Let A and B be unital C^*–algebras with faithful states ϕ_A and ϕ_B, respectively. Let*

$$(\mathfrak{A}, \phi) = (A, \phi_A) * (B, \phi_B)$$

by the free product of C^*–algebras. Then there is a projection of norm 1, E, from $\mathfrak{A}$ onto A, such that $\phi \circ E = \phi$.

Proof. Consider the projections of norm 1, $\mathrm{id}_A : A \to A$ and $\phi_B : B \to \mathbf{C}1 \subseteq B$. Let $E = \mathrm{id}_A * \phi_B$ be their free product.

□

§3. Some free product C*–algebras.

Theorem 3.1. *Let A and B be C^*–algebras with faithful states ϕ_A, respectively, ϕ_B. Consider the C^*–algebra reduced free product*

$$(\mathfrak{A}, \phi) = (A, \phi_A) * (B, \phi_B).$$

Suppose there is a partial isometry $v \in A$, whose range projection, $q = vv^*$, and domain projection, $p = v^*v$, are orthogonal, and such that, for some $0 < \lambda < 1$, v is in the spectral subspace of ϕ_A associated to λ^{-1}, namely, that $\phi_A(xv) = \lambda^{-1}\phi_A(vx)$ for every $x \in A$. Note this implies $\phi(q) = \lambda\phi(p) < \phi(p)$. Let

$$A_{00} = \mathbf{C}p + \mathbf{C}q + (1 - p - q)A(1 - p - q)$$

and let $\mathfrak{A}_{00} = C^*(A_{00} \cup B)$. Suppose that q is equivalent in the centralizer of the restriction of ϕ to $\mathfrak{A}_{00}$ to a subprojection of p, and that the centralizer of the restriction of ϕ to $q\mathfrak{A}_{00}q$ contains an abelian subalgebra on which ϕ is diffuse (i.e. an abelian subalgebra to which the restriction of ϕ is given by an atomless measure — see [**6**, 2.1]*). Suppose also that $p+q$ is full in $\mathfrak{A}$.*

Then $\mathfrak{A}$ is simple and purely infinite.

Proof. Since p is full in $\mathfrak{A}$, in order to show that $\mathfrak{A}$ is simple and purely infinite, it will suffice to show that $p\mathfrak{A}p$ is simple and purely infinite. By [**5**], ϕ is faithful on $\mathfrak{A}$.

By assumption there is a partial isometry, y, in the centralizer of $\phi|_{\mathfrak{A}_{00}}$, such that $yy^* = q$ and $p_1 \overset{\text{def}}{=} y^*y \leq p$. Also by assumption, the centralizer of the restriction of ϕ to $p_1\mathfrak{A}_{00}p_1$ contains a diffuse abelian subalgebra. Let $w = y^*v$. Then $w^*w = p$ and $ww^* = p_1 \leq p$.

Let $p_0 = p$ and for $n \geq 1$ let $p_n = w^n(w^*)^n$. Then w belongs to the spectral subspace of $\mathfrak{A}$ associated to λ^{-1}, and hence

$$\forall n \geq 0 \qquad \phi(p_n) = \lambda^n \phi(p). \tag{4}$$

Let

$$A_0 = pAp + (1-p)A(1-p).$$

Then A is generated by $A_0 \cup \{v\}$, because

$$A = pAp + pA(1-p) + (1-p)Ap + (1-p)A(1-p)$$

and

$$pA(1-p) = v^*vA(1-p) = v^*qA(1-p) \subseteq v^*(1-p)A(1-p).$$

Let $\mathfrak{A}_0 = C^*(A_0 \cup B)$. Then $\mathfrak{A} = C^*(\mathfrak{A}_0 \cup \{w\})$, and thus

$$p\mathfrak{A}p = C^*(p\mathfrak{A}_0 p \cup \{w\}).$$

In $p\mathfrak{A}p$, w is a proper isometry which, as we will see, is loosely speaking as free as it can be from $p\mathfrak{A}_0p$, with amalgamation over pA_0p.

Throughout the proof we will use a projection of norm 1, $E = E^{\mathfrak{A}}_{A_0} : \mathfrak{A} \to A_0$, which is hereby defined to be $E^{\mathfrak{A}}_{A_0} = E^{A}_{A_0} \circ E^{\mathfrak{A}}_{A}$, where $E^{\mathfrak{A}}_{A} : \mathfrak{A} \to A$ is the conditional expectation onto A obtained from Corollary 2.5, and where $E^{A}_{A_0} : A \to A_0$ is the conditional expectation onto A_0 defined by

$$E^{A}_{A_0}(a) = pap + (1-p)a(1-p).$$

Note that $\phi \circ E = \phi$.

Let Θ be the set of all

$$x = x_1 x_2 \cdots x_n \in \Lambda^{\circ}\big((p\mathfrak{A}_0p)^{\circ}, \{w^k \mid k \geq 1\} \cup \{(w^*)^k \mid k \geq 1\}\big) \tag{5}$$

such that whenever $2 \leq j \leq n-1$ and $x_j \in (p\mathfrak{A}_0p)^{\circ}$,

$$\begin{array}{ll} \text{if } x_{j-1} = w \text{ and } x_{j+1} = w^* & \quad \text{then } x_j \in p\mathfrak{A}_0p \ominus pA_0p \\ \text{if } x_{j-1} = w^* \text{ and } x_{j+1} = w & \quad \text{then } x_j \in p_1\mathfrak{A}_0p_1 \ominus \big(y^*(qA_0q)y\big). \end{array}$$

The restriction of E to $p\mathfrak{A}_0p$ provides a projection of norm 1 onto pA_0p, and $y^*E(y\cdot y^*)y$ provides a projection of norm 1 from $p_1\mathfrak{A}_0p_1$ onto $y^*(qA_0q)y$, so Lemma 2.2 applies. Notice that

$$\begin{aligned} waw^* &= wpapw^* \\ wA_0w^* &\subseteq p\mathfrak{A}_0p \\ w^*aw &= w^*p_1ap_1w, \\ w^*y^*(qA_0q)yw &\subseteq p\mathfrak{A}_0p \end{aligned}$$

and hence

$$\mathrm{span}(\{1\}\cup\Theta)=\mathrm{span}\Big(\{1\}\cup\Lambda^{\mathrm{o}}\big((p\mathfrak{A}_0p)^{\mathrm{o}},\{w^k\mid k\geq 1\}\cup\{(w^*)^k\mid k\geq 1\}\big)\Big)$$

is the $*$–algebra generated by $\mathfrak{A}_0\cup\{w\}$.

Like in [**4**], for $x\in\Theta$ let $t_j(x)$ be the number of w minus the number of w^* appearing in the first j letters of x. Thus, if $l(x)$ is the number of letters of x (i.e. $l(x)=n$ when x is as in (5)), after setting $t_0(x)=0$ we thus define

$$t_j(x)=\begin{cases} t_{j-1}(x) & \text{if the } j\text{th letter of } x \text{ is from } \mathfrak{A}_0^{\mathrm{o}} \\ t_{j-1}(x)+k & \text{if the } j\text{th letter of } x \text{ is } w^k \\ t_{j-1}(x)-k & \text{if the } j\text{th letter of } x \text{ is } (w^*)^k, \end{cases}$$

for each $1\leq j\leq l(x)$. We will use interval notation to denote subsets of the integers. Thus, for example,

$$\begin{gathered} [0,n] \text{ will mean } \{0,1,2,\ldots,n\} \\ [0,\infty) \text{ will mean } \{0,1,2,\ldots\} \\ (-\infty,0] \text{ will mean } \{\ldots,-2,-1,0\} \\ (-\infty,\infty) \text{ will mean } \mathbf{Z}. \end{gathered}$$

For every interval, I, of $\mathbf{Z}$ which contains 0, let

$$\Theta_I=\{x\in\Theta\mid t_{l(x)}=0 \text{ and } \forall 1\leq j\leq l(x),\, t_j(x)\in I\}.$$

Then $\mathrm{span}(\Theta_I\cup\{1\})$ is a $*$–subalgebra of $p\mathfrak{A}p$. Let $\mathfrak{A}_I=\overline{\mathrm{span}}(\Theta_I\cup\{1\})$.

There is an injective endomorphism, σ, of $\mathfrak{A}_{(-\infty,\infty)}$ given by $\sigma(a)=waw^*$. Since $p\mathfrak{A}p=C^*(\mathfrak{A}_{(-\infty,\infty)}\cup\{w\})$, it follows that $p\mathfrak{A}p$ is a quotient of the (universal) C*–algebra crossed product

$$\mathfrak{A}_{(-\infty,\infty)}\rtimes_\sigma\mathbf{N}.$$

We will use [**8**, Theorem 2.1(ii)] to prove that $\mathfrak{A}_{(-\infty,\infty)} \rtimes_\sigma \mathbf{N}$ is simple and purely infinite, which will imply that $p\mathfrak{A}p$ is simple and purely infinite. In particular, Claim 3.1a and Claim 3.1g below will show that the endomorphism σ satisfies the hypotheses of [**8**, Theorem 2.1(ii)].

Let $\widetilde{\mathfrak{A}}_{(-\infty,\infty)}$ be the C*–algebra inductive limit

$$\mathfrak{A}_{(-\infty,\infty)} \xrightarrow{\sigma} \mathfrak{A}_{(-\infty,\infty)} \xrightarrow{\sigma} \mathfrak{A}_{(-\infty,\infty)} \xrightarrow{\sigma} \cdots \to \widetilde{\mathfrak{A}}_{(-\infty,\infty)}$$

and for $n \geq 1$ consider the defining $*$–homomorphisms $\mu_n : \mathfrak{A}_{(-\infty,\infty)} \to \widetilde{\mathfrak{A}}_{(-\infty,\infty)}$ such that $\mu_{n+1} \circ \sigma = \mu_n$. There is an automorphism α of $\widetilde{\mathfrak{A}}_{(-\infty,\infty)}$ defined by $\alpha(\mu_n(a)) = \mu_n(\sigma(a))$.

Claim 3.1a. *For each $m \geq 1$, the automorphism α^m of $\widetilde{\mathfrak{A}}_{(-\infty,\infty)}$ is outer.*

Proof. If α^m is inner than $\alpha^m(a) = a$ for some $a \in \widetilde{\mathfrak{A}}_{(-\infty,\infty)}$, $a \geq 0$, $a \neq 0$. Let $p_{k-l} = \mu_l(\sigma^k(p))$. This coincides with the old definition of p_{k-l} if $k-l \geq 0$, and p_n is an approximate identity for $\widetilde{\mathfrak{A}}_{(-\infty,\infty)}$ as $n \to -\infty$. Thus $\|a - p_n a p_n\|$ can be made arbitrarily small by choosing n large and negative. But $\|a - p_n a p_n\| = \|\alpha^{km}(a - p_n a p_n)\| = \|a - p_{n+km} a p_{n+km}\|$. Since $\phi(p_{n+km} a p_{n+km}) \leq \|a\| \phi(p_{n+km})$ and since by (4) $\phi(p_{n+km})$ tends to 0 as $k \to \infty$, we may conclude that $\phi(a) = 0$. But since ϕ is faithful, this implies $a = 0$, which contradicts the choice of a.

This completes the proof of Claim 3.1a.

Claim 3.1b. *If $x \in \Theta_{(-\infty,\infty)}$ then $\phi(x) = 0$.*

Proof. Let $x = x_1 x_2 \cdots x_n \in \Theta_{(-\infty,\infty)}$. Rewrite each w appearing in x as vy^* and each w^* as yv^*. Now group together all occurances of y^*, y and letters from $(p\mathfrak{A}_0 p)^{\mathrm{o}}$ that are neighbors. The resulting object is either an element of $(p\mathfrak{A}_0 p)^{\mathrm{o}}$ or is equal to

$$u = u_1 u_2 \cdots u_m \in \Lambda^{\mathrm{o}}(\mathfrak{A}_0, \{v, v^*\}),$$

where whenever $2 \leq j \leq n-1$ and $u_j \in \mathfrak{A}_0$,

$$\begin{array}{rl} \text{if } u_{j-1} = v \text{ and } u_{j+1} = v & \quad \text{then } u_j \in p\mathfrak{A}_0 q \\ \text{if } u_{j-1} = v \text{ and } u_{j+1} = v^* & \quad \text{then } u_j \in p\mathfrak{A}_0 p \ominus pA_0 p \\ \text{if } u_{j-1} = v^* \text{ and } u_{j+1} = v & \quad \text{then } u_j \in q\mathfrak{A}_0 q \ominus qA_0 q \\ \text{if } u_{j-1} = v^* \text{ and } u_{j+1} = v^* & \quad \text{then } u_j \in q\mathfrak{A}_0 p. \end{array}$$

But

$$\mathfrak{A}_0 \subseteq \overline{\operatorname{span}}\big(\{1\} \cup \Lambda^{\mathrm{o}}(A_0^{\mathrm{o}}, B^{\mathrm{o}})\big)$$

and using the conditional expectation from $\mathfrak{A}_0$ onto A_0, we see that

$$p\mathfrak{A}_0q \cup (p\mathfrak{A}_0p \ominus pA_0p) \cup (q\mathfrak{A}_0q \ominus qA_0q) \cup q\mathfrak{A}_0p \subseteq \overline{\operatorname{span}}\big(\Lambda^{\mathrm{o}}(A_0^{\mathrm{o}}, B^{\mathrm{o}})\backslash A_0^{\mathrm{o}}\big).$$

Therefore, it will suffice to show that $\phi(z) = 0$ for every

$$z = z_1z_2\cdots z_s \in \Lambda^{\mathrm{o}}\big(\Lambda^{\mathrm{o}}(A_0^{\mathrm{o}}, B^{\mathrm{o}}), \{v, v^*\}\big)$$

which has the property that if $z_j \in \Lambda^{\mathrm{o}}(A_0^{\mathrm{o}}, B^{\mathrm{o}})$ and $2 \le j \le s-1$ then

$$z_j \in \Lambda^{\mathrm{o}}(A_0^{\mathrm{o}}, B^{\mathrm{o}})\backslash A_0^{\mathrm{o}}.$$

But since $A_0vA_0 \subseteq \ker\phi_A$, we see that $z \in \Lambda^{\mathrm{o}}(A^{\mathrm{o}}, B^{\mathrm{o}})$, and it then follows from the freeness of A and B that $\phi(z) = 0$.

This completes the proof of Claim 3.1b.

Claim 3.1c. *The subalgebras $w^*\mathfrak{A}_{(-\infty,0]}w$ and $\mathfrak{A}_{[0,\infty)}$ are free with amalgamation over pA_0p, (with respect to the restrictions of the conditional expectation E).*

Proof. Since $pA_0p = w^*y^*(qA_0q)yw$ we see that $pA_0p \subseteq w^*\mathfrak{A}_{(-\infty,0]}w$, and clearly $pA_0p \subseteq p\mathfrak{A}_{[0,\infty)}p$. In order to show freeness with amalgamation, (and refering to Lemma 2.2), it will suffice to show that $E(x) = 0$ whenever

$$x \in \Lambda^{\mathrm{o}}(\mathfrak{A}_{[0,\infty)} \ominus pA_0p, w^*\mathfrak{A}_{(-\infty,0]}w \ominus pA_0p). \tag{7}$$

Let $\Theta_{(-\infty,0^-]}$ be the set of all elements of $\Theta_{(-\infty,0]}$ which are words that begin with w^* and end with w. We will show that

$$w^*\mathfrak{A}_{(-\infty,0]}w \ominus pA_0p \subseteq \overline{\operatorname{span}}\Theta_{(-\infty,0^-]}. \tag{8}$$

Firstly, note that

$$w^*(\Theta_{(-\infty,0]}\backslash(p\mathfrak{A}_0p))w \subseteq \Theta_{(-\infty,0^-]}.$$

Now it will be enough to show that

$$w^*\mathfrak{A}_0w \ominus pA_0p \subseteq \Theta_{(-\infty,0^-]}.$$

But

$$\begin{aligned} w^*\mathfrak{A}_0w \ominus pA_0p &= w^*(p_1\mathfrak{A}_0p_1 \ominus wA_0w^*)w \\ &= w^*(p_1\mathfrak{A}_0p_1 \ominus y^*vA_0v^*y)w \\ &= w^*(p_1\mathfrak{A}_0p_1 \ominus y^*(qA_0q)y)w \subseteq \Theta_{(-\infty,0^-]}. \end{aligned}$$

We also see that

$$\mathfrak{A}_{[0,\infty)} \ominus pA_0p \subseteq \overline{\operatorname{span}}\big((\Theta_{[0,\infty)}\backslash(p\mathfrak{A}_0p)) \cup (p\mathfrak{A}_0p \ominus pA_0p)\big).$$

Hence, given x as in (7), in order to $E(x) = 0$ we may assume without loss of generality that

$$x \in \Lambda^{\mathrm{o}}\big(\Theta_{(-\infty,0^-]}, (\Theta_{[0,\infty)}\backslash(p\mathfrak{A}_0p)) \cup (p\mathfrak{A}_0p \ominus pA_0p)\big).$$

But then clearly

$$x \in (\Theta_{(-\infty,\infty)}\backslash\mathfrak{A}_0) \cup (p\mathfrak{A}_0p \ominus pA_0p).$$

If $x \in p\mathfrak{A}_0p \ominus pA_0p$ then by Lemma 2.2 $E(x) = 0$. Furthermore, using Claim 3.1b, that

$$p\mathfrak{A}_0p(\Theta_{(-\infty,\infty)}\backslash\mathfrak{A}_0) \subseteq \operatorname{span}(\Theta_{(-\infty,\infty)}\backslash\mathfrak{A}_0)$$

and Lemma 2.2, we see that if $x \in \Theta_{(-\infty,\infty)}\backslash\mathfrak{A}_0$ then $E(x) = 0$.

This completes the proof of Claim 3.1c.

Claim 3.1d. *The C^*–algebra $\mathfrak{A}_{(-\infty,0]}$ is simple.*

Proof. $\mathfrak{A}_{(-\infty,0]}$ is generated by $w^*\mathfrak{A}_{(-\infty,0]}w$ and $p\mathfrak{A}_0p$, which by Claim 3.1c are free with amalgamation over pA_0p. Now letting $\mathfrak{A}_0'$ be the C*–algebra generated by $B\cup(\mathbf{C}p+(1-p)A(1-p))$, using [**6**, 2.8] we see that $p\mathfrak{A}_0p$ is generated by pA_0p and $p\mathfrak{A}_0'p$, which are free with respect to the state ϕ (after rescaling). Hence we see that $\mathfrak{A}_{(-\infty,0]}$ is generated by $w^*\mathfrak{A}_{(-\infty,0]}w$ and $p\mathfrak{A}_0'p$ which are free (with amalgamation over the scalars $\mathbf{C}p$). But $w^*\mathfrak{A}_{(-\infty,0]}w \supseteq w^*(p_1\mathfrak{A}_{00}p_1)w$ and the restriction of ϕ to $w^*(p_1\mathfrak{A}_{00}p_1)w$ is just a rescaling of the restriction of ϕ to $p_1\mathfrak{A}_{00}p_1$. We saw earlier that the centralizer of the restriction of ϕ to $p_1\mathfrak{A}_{00}p_1$ has an abelian subalgebra which is diffuse with respect to ϕ, and hence so does the centralizer of the restriction of ϕ to $w^*(p_1\mathfrak{A}_{00}p_1)w$. Clearly $p\mathfrak{A}_0'p \neq \mathbf{C}$, so by [**6**, 3.2], $\mathfrak{A}_{(-\infty,0]}$ is simple.

Hence Claim 3.1d is proved.

Claim 3.1e. *For every $n \geq 0$, the C^*–algebra $\mathfrak{A}_{(-\infty,n]}$ is simple.*

Proof. We use induction on n. The case $n = 0$ holds by the previous claim. Assume $n \geq 1$. Now since $\mathfrak{A}_{(-\infty,0]}$ is simple, p_1 is full in $\mathfrak{A}_{(-\infty,0]}$, hence is full in $\mathfrak{A}_{(-\infty,n]}$. Therefore it will suffice to show that $p_1\mathfrak{A}_{(-\infty,n]}p_1$ is simple. But

$$p_1\mathfrak{A}_{(-\infty,n]}p_1 = ww^*\mathfrak{A}_{(-\infty,n]}ww^* \cong w^*\mathfrak{A}_{(-\infty,n]}w = \mathfrak{A}_{(-\infty,n-1]},$$

which is simple by inductive hypothesis.

Hence Claim 3.1e is proved.

Claim 3.1f. *Let $n \geq 0$ and $k \geq 1$ be integers. Then $p_{n+1}\mathfrak{A}_{(-\infty,n]}p_{n+1}$ and $\{p_{n+k}\}$ are free (with amalgamation over the scalars $\mathbf{C}p_{n+1}$) with respect to the state ϕ (after rescaling).*

Proof. The map $x \mapsto (w^*)^{n+1}xw^{n+1}$ is an isomorphism from $p_{n+1}\mathfrak{A}_{(-\infty,n]}p_{n+1}$ onto $w^*\mathfrak{A}_{(-\infty,0]}w$ which scales the state ϕ and which sends p_{n+k} to p_{k-1}. Hence it will suffice to show that $w^*\mathfrak{A}_{(-\infty,0]}w$ and $\{p_{k-1}\}$ are free (with amalgamation over the scalars $\mathbf{C}p$) with respect to ϕ (after rescaling). In light of Claim 3.1c, for this it will suffice to show that

$$E(p_{k-1}) = \frac{\phi(p_{k-1})}{\phi(p)}p. \tag{9}$$

However, $p_{k-1} \in C^*(B \cup (1-p-q)A(1-p-q) \cup \{v\})$, and $C^*((1-p-q)A(1-p-q) \cup \{v\})$ and B are free with respect to ϕ. Therefore,

$$E(p_{k-1}) \in E(C^*(B \cup (1-p-q)A(1-p-q) \cup \{v\})) = \mathbf{C}p + \mathbf{C}q + (1-p-q)A(1-p-q).$$

Now $p_{k-1} \leq p$ so $E(p_{k-1}) \in \mathbf{C}p$. But E preserves the state ϕ, so (9) follows.

This completes the proof of Claim 3.1f.

For the next claim, we will make use of the comparison theory for positive elements in a C*–algebra that was introduced by J. Cuntz [**2**], [**3**] (see also [**10**]). Recall that for positive elements, a and b of $\mathfrak{A}$, Cuntz defined $a \precsim b$ if there are $x_j \in \mathfrak{A}$ such that $\lim_{j\to\infty} x_j^* b x_j = a$. Recall also that $\precsim$ is a transitive relation.

Claim 3.1g. *Let D be a nonzero, hereditary C*–subalgebra of $\mathfrak{A}_{(-\infty,\infty)}$. Then there is a projection in D that is equivalent in $\mathfrak{A}_{(-\infty,\infty)}$ to p_n for some n.*

Proof. Let $h \in D$, $h \geq 0$, $\|h\| = 1$. Since $\bigcup_{n\geq 1}\mathfrak{A}_{(-\infty,n]}$ is dense in $\mathfrak{A}_{(-\infty,\infty)}$, for every $\epsilon > 0$ there is $n \in \mathbf{N}$ and $h_n \in \mathfrak{A}_{(-\infty,n]}$ such that $\|h_n\| = 1$ and $\|h - h_n\| < \epsilon$. Take $\epsilon < 1$ and let $f : [0,1] \to [0,1]$ be monotone increasing such that $f(1-\epsilon) = 0$ and $f(1) = 1$, and let $b = f(h_n)$. Then $b \geq 0$, $b \neq 0$. Since, by Claim 3.1e, $\mathfrak{A}_{(-\infty,n]}$ is simple,

$$b\big(\mathfrak{A}_{(-\infty,n]}\big)p_{n+1} \neq \{0\}.$$

Therefore, there is $a \in p_{n+1}\mathfrak{A}_{(-\infty,n]}p_{n+1}$ $a \geq 0$, $a \neq 0$ such that $a \precsim b$. By Claim 3.1f, a and p_{n+k} are free for every $k \geq 1$. Since $\lim_{k\to\infty}\phi(p_{n+k}) = 0$, using [**9**, 5.3], we see that for sufficiently large k we have $p_{n+k} \precsim a$. Thus $p_{n+k} \precsim b$ and hence there is $x \in \mathfrak{A}_{(-\infty,\infty)}$ such

that $\|x^*bx - p_{n+k}\| < \epsilon$. But

$$x^*bx \geq x^*b^{\frac{1}{2}}h_nb^{\frac{1}{2}}x \geq (1-\epsilon)x^*bx$$
$$\|x^*bx - x^*b^{\frac{1}{2}}h_nb^{\frac{1}{2}}x\| \leq \epsilon\|x^*bx\| < \epsilon(1+\epsilon)$$
$$\|x^*b^{\frac{1}{2}}h_nb^{\frac{1}{2}}x - x^*b^{\frac{1}{2}}hb^{\frac{1}{2}}x\| \leq \|h_n - h\|\ \|x^*bx\| < \epsilon(1+\epsilon)$$
$$\|x^*b^{\frac{1}{2}}hb^{\frac{1}{2}}x - p_{n+k}\| < \epsilon(3+2\epsilon).$$

By standard arguments, taking ϵ small enough we find a projection in $\overline{h\mathfrak{A}_{(-\infty,\infty)}h}$ that is equivalent to p_{n+k}.

Hence Claim 3.1g is proved.

Now using Claim 3.1a, Claim 3.1g and [**8**, Theorem 2.1(ii)] completes the proof of the proposition.

□

Now we give a list (by no means complete) of examples where the above proposition can be applied. For $0 < \lambda \leq 1$, with the symbol ψ_λ we denote the state on $M_2(\mathbf{C})$,

$$\psi_\lambda(\cdot) = \mathrm{Tr}_2\left(\cdot\begin{pmatrix} \frac{1}{1+\lambda} & 0 \\ 0 & \frac{\lambda}{1+\lambda}\end{pmatrix}\right).$$

Examples 3.2. *In each of the following cases, Theorem 3.1 applies, showing that the reduced free product C^*–algebra $\mathfrak{A}$ is simple and purely infinite.*

(i) *Let $0 < \lambda < \mu \leq 1$ and let*

$$(\mathfrak{A}, \phi) = (M_2(\mathbf{C}), \psi_\lambda) * (M_2(\mathbf{C}), \psi_\mu).$$

(ii) *Let $0 < \lambda < 1$ and let*

$$(\mathfrak{A}, \phi) = (M_2(\mathbf{C}), \psi_\lambda) * (C([0,1]), \int \cdot\, dt).$$

(ii) *Let*

$$(\mathfrak{A}, \phi) = (A_1 \otimes F, \phi_{A_1} \otimes \phi_F) * (B, \phi_B)$$

where A_1 is any C^*–algebra, ϕ_{A_1} is any faithful state on A_1, F is a finite dimensional C^*–algebra, ϕ_F is a nontracial, faithful state on F and the centralizer of ϕ_B contains a unital, abelian C^*–subalgebra on which ϕ_B is diffuse.

Proof. We first consider case (i). In the first copy of $M_2(\mathbf{C})$, let $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Let p', q' and v' be the same but in the second copy of $M_2(\mathbf{C})$. Then v is a partial isometry as required in Theorem 3.1. Moreover, p, q, p' and q' are all in the centralizer of ϕ and the C*–algebra generated by $\{p,q,p',q'\}$ is isomorphic to the free product, (in the notation of [**6**]),

$$(\underset{\frac{1}{1+\lambda}}{\overset{p}{\mathbf{C}}} \oplus \underset{\frac{\lambda}{1+\lambda}}{\overset{q}{\mathbf{C}}}) * (\underset{\frac{1}{1+\mu}}{\overset{p'}{\mathbf{C}}} \oplus \underset{\frac{\mu}{1+\mu}}{\overset{q'}{\mathbf{C}}}),$$

which can be described by refering to [**6**, 2.7]. Thus

$$C^*(\{p,q,p',q'\}) \cong \underset{\frac{1}{1+\lambda}-\frac{1}{1+\mu}}{\overset{p\wedge q'}{\mathbf{C}}} \oplus C([a,b],M_2(\mathbf{C})) \oplus \underset{\frac{1}{1+\lambda}-\frac{\mu}{1+\mu}}{\overset{p\wedge p'}{\mathbf{C}}},$$

for some $0 < a < b < 1$, where the trace on $C([a,b],M_2(\mathbf{C}))$ is induced by a measure on $[a,b]$ having no atoms, and with

$$p = 1 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 1$$
$$q = 0 \oplus \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0.$$

Hence q is equivalent to a subprojection of p in $C^*(\{p,q,p',q'\})$ and $qC^*(\{p,q,p',q'\})q$ contains a diffuse abelian subalgebra, so the hypotheses of Theorem 3.1 are fulfilled.

It is clear that case (ii) follows from case (iii). In order to prove that Theorem 3.1 applies in case (iii), note that since ϕ_F is nontracial, there is a partial isometry, $v \in F$, with $p \overset{\text{def}}{=} v^*v$ and $q \overset{\text{def}}{=} vv^*$ orthogonal and minimal projections in F such that $\phi(q) < \phi(p)$. Moreover, v is in the spectral subspace of $\phi_A \otimes \phi_F$ associated to $\lambda^{-1} = \phi(p)/\phi(q)$. Let D be a unital abelian C*–subalgebra of the centralizer of ϕ_B on which ϕ_B is diffuse. Consider $C^*(\{p,q\} \cup D)$. This is isomorphic to the reduced free product of abelian C*–algebras

$$(\overset{p}{\mathbf{C}} \oplus \overset{q}{\mathbf{C}} \oplus \overset{1-p-q}{\mathbf{C}}) * (D, \phi|_D)$$

or to

$$(\overset{p}{\mathbf{C}} \oplus \overset{q}{\mathbf{C}}) * (D, \phi|_D),$$

depending on whether $p+q=1$ or not. By [**6**, 5.3], $C^*(\{p,q\} \cup D)$ is simple and hence $p+q$ is full in it. By [**7**, 4.6(i)], there is a unitary $u \in D$ such that $\phi(u) = 0$. Then p and u^*pu are free. If $p+q=1$ then $C^*(\{1,p,u^*pu\})$ is isomorphic to the free product

$$(\underset{\frac{1}{1+\lambda}}{\overset{p}{\mathbf{C}}} \oplus \underset{\frac{\lambda}{1+\lambda}}{\overset{q}{\mathbf{C}}}) * (\underset{\frac{1}{1+\lambda}}{\overset{u^*pu}{\mathbf{C}}} \oplus \underset{\frac{\lambda}{1+\lambda}}{\overset{u^*qu}{\mathbf{C}}}).$$

Refering again to [**6**, 2.7] we see that

$$C^*(\{p,u^*pu\}) \cong \{f:[0,b]\to M_2(\mathbf{C}) \mid f \text{ continuous and } f(0) \text{ diagonal }\} \oplus \underset{\frac{1-\lambda}{1+\lambda}}{\overset{p\wedge u^*pu}{\mathbf{C}}},$$

for some $0<b<1$,where the trace corresponds to a measure on $[0,b]$ having no atoms, and with

$$\begin{aligned} p &= \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \oplus 1 \\ q &= \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \oplus 0 \\ u^*pu &= \begin{pmatrix} t & \sqrt{t(1-t)} \\ \sqrt{t(1-t)} & 1-t \end{pmatrix} \oplus 1 \\ u^*qu &= \begin{pmatrix} 1-t & -\sqrt{t(1-t)} \\ -\sqrt{t(1-t)} & t \end{pmatrix} \oplus 0. \end{aligned}$$

Thus $qC^*(\{p,u^*pu\})q$ contains a diffuse abelian subalgebra and u^*qu is equivalent in $C^*(\{p,u^*pu\})$ to a subprojection of p, hence q is equivalent in $C^*(\{p,q\}\cup D)$ to a subprojection of p. So in the case $p+q=1$ we are done. But if $p+q\neq 1$ then by [**6**, 2.8], $(p+q)C^*(\{p,q\}\cup D)(p+q)$ is isomorphic to the free product of $(\mathbf{C}p+\mathbf{C}q)$ and $(p+q)C^*(\{p+q\}\cup D)(p+q)$, while by [**6**, 3.5], the latter algebra has a diffuse abelian subalgebra. Therefore, the analysis we just did for the case $p+q=1$ applies to the free product of $(\mathbf{C}p+\mathbf{C}q)$ and $(p+q)C^*(\{p+q\}\cup D)(p+q)$, showing that $qC^*(\{p,u^*pu\})q$ contains a diffuse abelian subalgebra and q is equivalent in $C^*(\{p,q\}\cup D)$ to a subprojection of p.

$\square$

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